EXERCISE 18.4

1. (a) Find the greatest value of the function $2 + x - 3x^2$.
   (b) Find the greatest value of the function xe^{-x}.

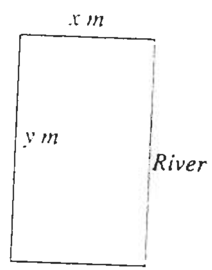
2. (a) Find the least value of the function $8 - 3x + 2x^2$.
   (b) Find the least value of the function $x - \log_e x$.
   (c) Find the least value of the function $4 - \dfrac{2}{1 + x^2}$.

3. Find the range of the function $x \mapsto \dfrac{x}{1 + x^2}$.

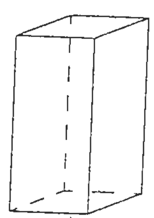
4. Two real numbers x and y are such that $x + y = 21$. Find the value of x for which
   (a) the product, xy, is a maximum.
   (b) the product xy^3 is a maximum.

5. If $x + y = 12$, find the minimum value that $x^2 + y^2$ can have.

6. A farmer wishes to fence off a rectangular paddock using an existing stretch of a river as one side. The total length of wiring available is 100 m.
   Let x m and y m denote the length and width of this rectangular paddock respectively, and let A m^2 denote its area.
   (a) Obtain an expression for y in terms of x.
   (b) Find an expression for A in terms of x, stating any restrictions on x.
   (c) Determine the dimensions which will maximize the area enclosed by this rectangle.



7. A closed rectangular box with square ends is to be constructed in such a way that its total surface area is 400 cm^2. Let x cm be the side length of the ends and y cm its height.
   (a) Obtain an expression for y in terms of x, stating any restrictions on x.
   (b) Find the largest possible volume of all such boxes.



8. A barrel is being filled with water in such a way that the volume of water, V ml, in the barrel after time t seconds is given by

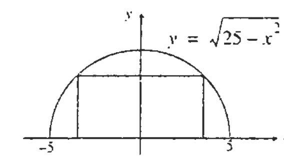
$$V(t) = \frac{2}{3}\left(20t^2 - \frac{1}{6}t^3\right),\ 0 \le t \le 120$$

   (a) Find the rate of flow into the barrel after 20 seconds.
   (b) When will the rate of flow be greatest?
   (c) Sketch the graph of $V(t),\ 0 \le t \le 120$.

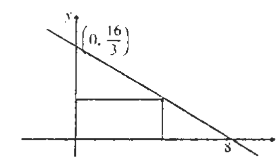
9. The total cost, \$$C$, for the production of x items of a particular product is given by the linear relation $C = 600 + 20x,\ 0 \le x \le 100$, whilst its total revenue, \$$R$, is given by $R = x(100 - x),\ 0 \le x \le 100$.
   (a) Sketch the graph of the cost function and revenue function on the same set of axes.
   (b) Determine the break–even points on your graph.
   (c) For what values of x will the company be making a positive profit?
   (d) Find an expression that gives the profit made in producing x items of the product, and hence determine the maximum profit.

10. Find the points on the graph of $y = 9 - x^2$ that are closest to the point (0, 3).

11. A rectangle is bounded by the semi–circle with equation $y = \sqrt{25 - x^2},\ -5 \le x \le 5$ and the x–axis. Find the dimensions of the rectangle having the largest area.



12. A rectangle is bounded by the positive x-axis the positive y-axis and the line with equation $y = \frac{2}{3}(8 - x)$.
   Find the dimensions of the rectangle having the largest area.



13. A certificate is to be printed on a page having an area of 340 cm^2. The margins at the top and bottom of the page are to be 2cm and, on the sides, 1cm.
   (a) If the width of the page is x cm, show that the area, A cm^2 where printed material is to appear is given by

$$A = 348 - \frac{680}{x} - 4x.$$

   (b) Hence, determine the maximum area of print.

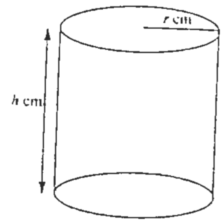
14. Find the minimum value of the sum of a positive integer and its reciprocal.

15. A cylinder is to have a surface area of $20\pi\ \text{cm}^2$. Determine the dimensions of the cylinder which will have the largest volume.

16. A right circular cylinder of radius r cm and height h cm is to have a fixed volume of $30\ \text{cm}^3$.

(a) Show that the surface area, $A\ \text{cm}^2$ of such a cylinder is given by $A = 2\pi r\left(r + \dfrac{30}{\pi r^2}\right)$.

(b) Determine the value of r that will yield the minimum surface area.

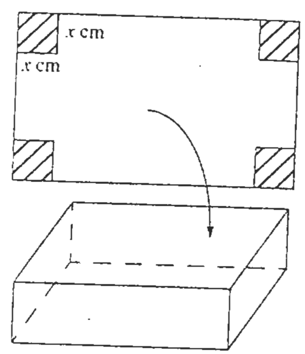


17. A rectangular container is made by cutting out squares from the corners of a 25cm by 40cm rectangular sheet of metal and folding the remaining sheet to form the container.

(a) If the squares that are cut out are x cm in length, show that the volume, $V\ \text{cm}^3$ of the container is given by

$V = x(25-2x)(40-2x),\ 0 \le x \le \frac{25}{2}$

(b) What size squares must be cut out in order to maximize the volume of the container?



18. A right–circular cone of radius r cm contains a sphere of radius 12cm.

(a) If the height of the cone is h cm, express h in terms of r.

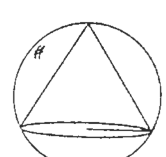
(b) If $V\ \text{cm}^3$ denotes the volume of the cone, find an expression for V in terms of r.

(c) Find the dimensions of the cone with the smallest volume.

19. For a closed cylinder of radius r cm and height h cm, find the ratio $r{:}h$ which will produce the smallest surface area for a fixed volume.

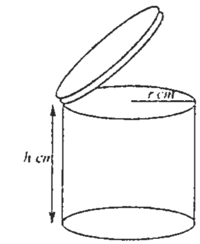
20. For an open cylinder of radius r cm and height h cm, find the ratio $r{:}h$ which will produce the smallest surface area for a fixed volume.

21. Find the height of a right–circular cone which can be inscribed in a sphere of radius 1 m, if this cone is to have the largest possible volume.

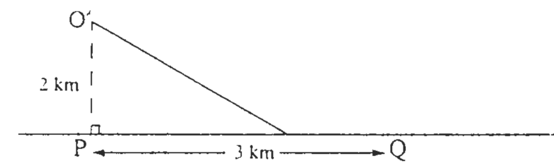


22. A piece of wire 30 cm long is cut into 2 pieces. One of the pieces is bent into a square whilst the other is bent into a circle. Find the ratio of the side length of the square to the radius of the circle which provides the smallest area sum.

23. A cylindrical biscuit tin having a lid 1cm deep is to have a capacity of $144\pi\ \text{cm}^3$.
If the cylinder is of radius r cm and height h cm, find the ratio $r{:}h$ which will give the smallest surface area.

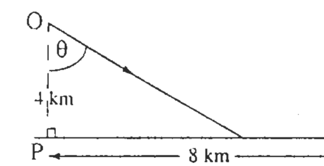


24. The last leg of a triathlon requires that you get from a point O, 2km from the nearest point P on a straight beach to a point Q, 3km down the coast.



You may swim to any point on the beach and then run the rest of the way to point Q. If you can swim at a rate of 2km/hr and run at 5km/hr, where should you land on the beach in order to reach point Q in the least possible time?

25. A person in a boat 4km from the nearest point P on a straight beach, wishes to get to a point Q 8km along the beach from P.
The person rows in a straight line to some point on the shore at a constant rate of 5km/h. Once on the beach the person walks towards Q at a steady rate of 6km/hr.



(a) Show that the total time taken for the trip is

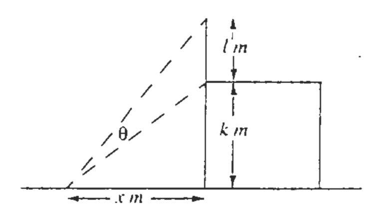
$$T(\theta) = \frac{4\sec\theta}{5} + \frac{8-4\tan\theta}{6},\ \theta \in \left[0, \frac{\pi}{2}\right) \text{ and } \tan\theta \le 2$$

(b) Where should the person land so that the trip takes the least amount of time?

26. A mast l metres tall erected on a building k metres tall, subtends an angle θ at a point on the ground x metres from the base.

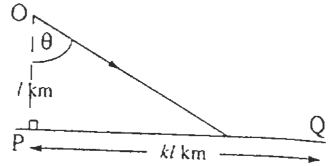
(a) Find an expression for $\tan\theta$ in terms of x.

(b) Find the value of x that maximizes the the value of θ.



[Hint: As $\theta \in \left[0, \frac{\pi}{2}\right)$, θ is a maximum when $\tan\theta$ is a maximum.]

27. A person in a boat l km from the nearest point P on a straight beach, wishes to get to a point Q kl km along the beach from P.
The person rows in a straight line to some point on the shore at a constant rate of v km/h. Once on the beach the person walks towards Q at a steady rate of u km/hr where ($v < u$).



(a) Show that the total time taken for the trip is

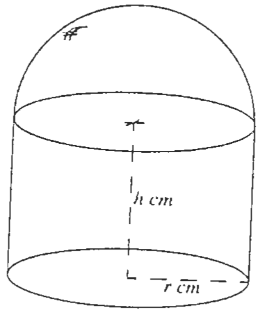
$$T(\theta) = \frac{l\sec\theta}{v} + \frac{kl - l\tan\theta}{u},\ \theta \in \left[0, \frac{\pi}{2}\right) \text{ and } \tan\theta \le k$$

(b) Show that if the person is to reach point Q in the least possible time, then

$\sin\theta = \frac{v}{u}$ where $k \ge c$, c being a particular constant.

(c) What would happen if $k < c$?

28. A closed tin is to be constructed as shown in the diagram below. It is made up of a cylinder of height h cm and radius base r cm which is surmounted by a hemispherical cap.



(a) Find an expression in terms of r and h for

i. its volume, $V\ cm^3$

ii. its surface area, $A\ cm^2$.

(b) Given that $V = \pi k^3, k > 0$, show that its surface area is given by

$$A = 2\pi k^3\frac{1}{r} + \frac{5\pi}{3}r^2.$$

(c) Find $r{:}h$ for A to be a minimum.

29. A ladder is to be carried horizontally around a corner from a corridor a m wide into a corridor b m wide. What is the maximum length that the ladder can be?